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Asymptotic and Oscillatory Behavior of nth Order Forced Functional Differential Equations

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Some new oscillation criteria for the forced functional differential equation $x^{(n)}(t)+a(t)f(x[q(t)])=e(t)$ are established. © 1989 Academic Press, Inc.

1. INTRODUCTION

This paper is concerned with the asymptotic and oscillatory behavior of solutions of the forced functional equation

$$x^{(n)}(t)+a(t)f(x[q(t)])=e(t), \tag{1}$$

where a, e, q: $[t_0, \infty)\to R=(-\infty, \infty)$, f: $R\to R$ are continuous, $a(t)>0$, $\lim_{t\to\infty} q(t)=\infty$, and $xf(x)>0$ for $x\neq 0$.

By a solution of Eq. (1) we mean a function x: $[T_x, \infty)\to R$ which satisfies Eq. (1) for all sufficiently large t and $\sup\{|x(t)|: t\geqslant T\}>0$ for any $T\geqslant T_x$. Such a solution is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

The behavioral properties of Eq. (1) and/or related equations have been studied by many authors; we mention here the works of Dahiya and Akinyele [1], the present authors [3–5], Kartsatos [6–8], and Žilina

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[10], where condition of monotonicity on the function f appearing in Eq. (1) is needed. Also, the reader is referred to the results of Mahfoud [9], who replaces the condition of monotonicity imposed on f by requiring the function f to be either locally of bounded variation or else continuously differentiable.

In [1], Dahiya and Akinyele gave some criteria for the asymptotic and oscillatory behavior of solutions of linear retarded or advanced differential equations of the form of Eq. (1). However, these results cannot be applied to differential Eq. (1) when:

(a) f is a nonlinear function (either differentiable or not), e.g., $f(x)=|x|^{\alpha}\operatorname{sgn} x$, $\alpha>0$, or $x/(1+x^2)$, or $xe^{\sin x}$, ..., etc.;

(b) q is of mixed type, e.g., $q(t)=t\pm\sin t$, ..., etc.;

(c) $a(t)=O(t^{-n})$ and $q(t)=O(t)$.

Therefore, the purpose of this paper is to establish some new criteria for the asymptotic and oscillatory behavior of solutions of Eq. (1). These theorems can be applied to a class of both forced and unforced differential equations of the form of Eq. (1). It is shown that the behavioral properties of the unforced equations are maintained under the effect of certain forcing terms. Our results can be applied to differential equations when results of Dahiya and Akinyele [1] fail.

2. Main Results

The following notations will be used throughout this paper:

$$R_\alpha=(-\infty,-\alpha]\cup[\alpha,\infty),\ \alpha\geqslant 0,$$

$$C(R)=\{f\colon R\to R\mid f \text{ is continuous and } xf(x)>0 \text{ if } x\neq 0\},$$

$$C^1(R)=\{f\in C(R)\mid f \text{ is continuously differentiable in } R_\alpha\}, \text{ and}$$

$$C_p(R_\alpha)=\{f\in C(R)\mid f \text{ is of bounded variation on every } [a,b]\subset R\}.$$

The following two lemmas will be needed in the proofs of our results. The first lemma can be found in [10] and the second appeared in [9].

Lemma 1. *Let u be a positive and n-times differentiable function on an interval $[t_0,\infty)$ and let*

$$u^{(n)}(t)\leqslant 0 \qquad for \quad t\in[t_0,\infty).$$

Then there exist $t_u\in[t_0,\infty)$ and an integer $l\in\{0,1,...,n-1\}$ such that $n+l$ is odd and

(i) $u^{(k)}(t)>0$ *for* $t\in[t_u,\infty)$ $(k=0,1,...,l-1)$,

(ii) $(-1)^{l+k} u^{(k)}(t) > 0$ *for* $t \in [t_u, \infty)$ $(k = l, l+1, ..., n-1)$,

(iii) $(t - t_u)|u^{(l-k)}(t)| \leqslant (1+k)|u^{(l-k-1)}(t)|$ *for* $t \in [t_u, \infty)$ $(k = 0, 1, ..., l-1)$, $1 \leqslant l \leqslant n-1$.

LEMMA 2. *Suppose* $\alpha \geqslant 0$ *and* $f \in C(R)$. *Then* $f \in C_p(R_\alpha)$ *if and only if* $f(x) = g(x)\, h(x)$ *for all* $x \in R_\alpha$, *where* $g\colon R_\alpha \to (0, \infty)$, *nondecreasing on* $(-\infty, -\alpha)$ *and nonincreasing on* (α, ∞) *and* $f\colon R_\alpha \to R$ *and nondecreasing on* R_α.

THEOREM 1. *Suppose* $f \in C(R_\alpha)$, $\alpha > 0$, *and let* g *and* h *be a pair of continuous components of* f *with* h *being the nondecreasing one,*

$$\frac{h(x)}{x} \geqslant \gamma > 0 \qquad \text{for} \quad x \neq 0. \tag{2}$$

Suppose there exist continuous functions η, σ: $[t_0, \infty) \to R$ *such that*

$$\alpha \leqslant \sigma(t) \leqslant \min\{t, q(t)\} \qquad \text{and} \qquad \lim_{t \to \infty} \sigma(t) = \infty; \tag{3}$$

$$\eta(t) \text{ is oscillatory, } \eta^{(n)}(t) = e(t), \text{ and } \lim_{t \to \infty} \eta^{(i)}(t) = 0,\ i = 0, 1, ..., n-1. \tag{4}$$

If

$$\limsup_{t \to \infty} \left[\frac{1}{t}\int_T^t s\sigma^{n-1}(s)\, a(s)\, g(kq^{n-1}(s))\, ds \right.$$
$$\left. + t\int_t^\infty \frac{\sigma^{n-1}(s)}{s} a(s)\, g(kq^{n-1}(s))\, ds\right] > \frac{2(n-1)!}{\gamma}, \tag{5}$$

for every $k \geqslant 1$ *and* $T \geqslant t_0$, *then, for* n *even, Eq.* (1) *is oscillatory, while, for* n *odd, every solution* $x(t)$ *of Eq.* (1) *is either oscillatory or* $x^{(i)}(t) \to 0$ *as* $t \to \infty$, $i = 0, 1, ..., n-1$.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Without loss of generality, we assume that $x(t) \neq 0$ for $t \geqslant t_0$. Furthermore, we suppose that $x(t) > 0$ and $x[q(t)] > 0$ for $t \geqslant t_0$, since the substitution $u = -x$ transforms Eq. (1) into an equation of the same form subject to the assumptions of the theorem. Consider the function

$$x(t) = y(t) + \eta(t), \qquad t \geqslant t_0$$

then from Eq. (1) we have

$$y^{(n)}(t) = -a(t)\, f(x[q(t)]) < 0 \qquad \text{for} \quad t \geqslant t_0, \tag{6}$$

so that $y^{(n)}(t)$ is eventually negative for $t \geq t_0$. Hence $y^{(k)}(t)$, $k=0, 1, ..., n-1$, are monotone and one-signed for all sufficiently large t, say $t \geq t_1 \geq t_0$. If $y(t)<0$ for $t \geq t_1$, then $x(t)<\eta(t)$ for all $t \geq t_1$, which shows that $x(t)$ takes on negative values for arbitrarily large t. But this contradicts the assumption that $x(t)>0$ for $t \geq t_0$, and so we must have $y(t)>0$ for $t \geq t_1$. Notice next that the hypotheses of Lemma 1 are satisfied on $[t_1, \infty)$, which implies that there exists a $t_2 \geq t_1$ such that

$$|\dot{y}(t)|>0, \; y^{(n-1)}(t)>0, \quad \text{and} \quad y^{(n)}(t) \leq 0 \qquad \text{for all} \quad t \geq t_2$$

and hence $y^{(n-1)}(t)$ is nonincreasing for all $t \geq t_2$. Thus there exists a $c>0$ such that

$$y^{(n-1)}(t) \leq c \qquad \text{for all} \quad t \geq t_2.$$

By successive integrations from t_2 to t we conclude that

$$y(t) \leq \frac{c}{(n-1)!}(t-t_2)^{n-1} + \cdots + y(t_2) \qquad \text{for all} \quad t \geq t_2.$$

Choose a $t_3 \geq t_2$ and a $c_1>0$ so that

$$y(t) \leq c_1 t^{n-1} \qquad \text{for every} \quad t \geq t_3.$$

By (3) and (4), there exists a $t_4 \geq t_3$ such that $q(t) \geq \sigma(t) \geq t_3$ for all $t \geq t_4$ and

$$y[q(t)] \leq kq^{n-1}(t) \qquad \text{and} \qquad x[q(t)] \leq kq^{n-1}(t) \tag{7}$$

for some $k \geq 1$ and all $t \geq t_4$.

Next, we distinguish the following two cases according to the values of the integer l in Lemma 1.

Case 1. $l \in \{1, 2, ..., n-1\}$. Clearly $\dot{y}(t)>0$ for $t \geq t_4$ and hence

$$y[q(t)] \geq y[\sigma(t)] \geq \alpha > 0 \qquad \text{for all} \quad t \geq t_4. \tag{8}$$

As

$$f(x) = g(x)\,h(x) \qquad \text{for all} \quad x \geq \alpha,$$

then

$$f(x)[q(t)]) = g(x[q(t)])\,h(x[q(t)]).$$

By (7) we have

$$f(x[q(t)]) \geq g(kq^{n-1}(t))\,h(x[q(t)]) \qquad \text{for all} \quad t \geq t_4. \tag{9}$$

Recall that $x[q(t)] = y[q(t)] + \eta[q(t)]$. Since $y(t)$ is positive and increasing, and $\eta(t) \to 0$ as $t \to \infty$, there exists a $t_5 \geqslant t_4$ such that

$$x[q(t)] \geqslant \tfrac{1}{2} y[q(t)] \qquad \text{for all} \quad t \geqslant t_5. \tag{10}$$

Using (8) and (10) in (9) we get

$$f(x[q(t)]) \geqslant g(kq^{n-1}(t))\, h(\tfrac{1}{2} y[\sigma(t)]) \qquad \text{for all} \quad t \geqslant t_5. \tag{11}$$

By (2) and (11) and the fact that h is nondecreasing we obtain

$$y^{(n)}(t) \leqslant -\frac{\gamma}{2}\, a(t)\, g(kq^{n-1}(t))\, y[\sigma(t)] \qquad \text{for every} \quad t \geqslant t_5. \tag{12}$$

Now, by applying Taylor's formula with remainder, we get

$$\begin{aligned} y^{(l)}(t) = {} & \sum_{j=0}^{n-l-1} (-1)^j \frac{(s-t)^j}{j!}\, y^{(l+j)}(s) \\ & + (-1)^{n-l} \int_t^s \frac{(u-t)^{n-l-1}}{(n-l-1)!}\, y^{(n)}(u)\, du, \qquad s \geqslant t \geqslant t_5. \end{aligned}$$

Using (i) and (ii) of Lemma 1 and (12) we have

$$y^{(l)}(t) \geqslant \int_t^s \frac{\gamma}{2} \frac{(u-t)^{n-l-1}}{(n-l-1)!}\, a(u)\, g(kq^{n-1}(u))\, y[\sigma(u)]\, du, \qquad s \geqslant t \geqslant t_5.$$

As $s \to \infty$, we obtain

$$y^{(l)}(t) \geqslant \int_t^\infty \frac{\gamma}{2} \frac{(u-t)^{n-l-1}}{(n-l-1)!}\, a(u)\, g(kq^{n-1}(u))\, y[\sigma(u)]\, du, \qquad t \geqslant t_5. \tag{13}$$

By (iii) of Lemma 1, there exists a $t_6 \geqslant t_5$ such that $\sigma(t) > t_5$ and

$$y[\sigma(t)] \geqslant \frac{[\sigma(t) - t_5]^{l-1}}{l!}\, y^{(l-1)}[\sigma(t)] \qquad \text{for every} \quad t \geqslant t_6.$$

Thus,

$$y^{(l)}(t) \geqslant \int_t^\infty \frac{\gamma}{2} \frac{(u-t)^{n-l-1}}{(n-l-1)!} \frac{[\sigma(u) - t_5]^{l-1}}{l!}\, a(u)\, g(kq^{n-1}(u))\, y^{(l-1)}[\sigma(u)]\, du. \tag{14}$$

Integrating (14) from T to t, $t > T \geqslant t_6$, we get

$$y^{(l-1)}(t) \geqslant y^{(l-1)}(T) + \int_T^t \frac{\gamma}{2} \frac{(u-T)^{n-l}}{(n-l)!} \frac{[\sigma(u)-t_5]^{l-1}}{l!} a(u)\, g(kq^{n-1}(u))$$
$$\times y^{(l-1)}[\sigma(u)]\, du + (t-T) \int_t^\infty \frac{\gamma}{2} \frac{(u-T)^{n-l-1}}{(n-l)!}$$
$$\times \frac{[\sigma(u)-t_5]^{l-1}}{l!} a(u)\, g(kq^{n-1}(u))\, y^{(l-1)}[\sigma(u)]\, du.$$

Choose $T_1 \geqslant T$ such that $\sigma(u) \geqslant T$ for $u \geqslant T_1$. Then for $t \geqslant T_1$ we have

$$y^{(l-1)}(t) \geqslant \frac{\gamma}{2(n-l)!\, l!} \left[\int_{T_1}^t (u-T)^{n-l} [\sigma(u)-T]^{l-1} a(u)\, g(kq^{n-1}(u)) \right.$$
$$\times y^{(l-1)}[\sigma(u)]\, du + (t-T) \int_t^\infty (u-T)^{n-l-1} [\sigma(u)-T]^{l-1}$$
$$\left. \times a(u)\, g(kq^{n-1}(u))\, y^{(l-1)}[\sigma(u)]\, du \right]. \tag{15}$$

It follows from (iii) of Lemma 1 that the function $y^{(l-1)}(t)/(t-T)$ is non-increasing for $t > T$. So we have

$$y^{(l-1)}[\sigma(u)] \geqslant \frac{\sigma(u)-T}{t-T}\, y^{(l-1)}(t) \qquad \text{for} \quad t \geqslant u > T_1,$$

and

$$y^{(l-1)}[\sigma(u)] \geqslant \frac{\sigma(u)-T}{u-T}\, y^{(l-1)}(u) \qquad \text{since} \quad \sigma(u) \leqslant u.$$

Then (15) yields

$$\frac{2(n-l)!\, l!}{\gamma} \geqslant \frac{1}{t-T} \int_{T_1}^t (u-T)^{n-l} [\sigma(u)-T]^l a(u)\, g(kq^{n-1}(u))\, du$$
$$+ (t-T) \int_t^\infty (u-T)^{n-l-2} [\sigma(u)-T]^l a(u)\, g(kq^{n-1}(u))\, du.$$

Since

$$(u-T)^{n-l} [\sigma(u)-T]^l \geqslant (u-T)[\sigma(u)-T]^{n-1}$$
$$\text{for} \quad 0 \leqslant l \leqslant n-1 \text{ and } u \geqslant T_1,$$

we have

$$\frac{2(n-1)!}{\gamma} \geqslant \frac{2(n-l)!\,l!}{\gamma} \geqslant \frac{1}{t-T}\int_{T_1}^{t} (u-T)[\sigma(u)-T]^{n-1}\, a(u)\, g(kq^{n-1}(u))\, du$$
$$+ (t-T)\int_{t}^{\infty} \frac{[\sigma(u)-T]^{n-1}}{u-T}\, a(u)\, g(kq^{n-1}(u))\, du,$$

which contradicts condition (5).

Case 2. $l=0$. Then n is odd, and condition (5) implies that

$$\int^{\infty} s^{n-1} a(s)\, ds = \infty.$$

Now it is not hard to note that $\lim_{t\to\infty} x^{(i)}(t) = 0$ for $i = 0, 1, ..., n-1$. This completes the proof of the theorem.

The following corollaries are immediate.

COROLLARY 1. *Let* $g(x)=1$ *(i.e.,* $f(x)=h(x)$*) and let condition* (5) *in Theorem* 1 *be replaced by*

$$\limsup_{t\to\infty} \left[\frac{1}{t}\int_{T}^{t} s\sigma^{n-1}(s)\, a(s)\, ds + t\int_{t}^{\infty} \frac{\sigma^{n-1}(s)}{s}\, a(s)\, ds\right] > \frac{2(n-1)!}{\gamma}, \tag{16}$$

then the conclusion of Theorem 1 *holds.*

COROLLARY 2. *Let* $e(t)=0$ *and let condition* (5) *in Theorem* 1 *be replaced by*

$$\limsup_{t\to\infty} \left[\frac{1}{t}\int_{T}^{t} s\sigma^{n-1}(s)\, a(s)\, g(kq^{n-1}(s))\, ds + t\int_{t}^{\infty} \frac{\sigma^{n-1}(s)}{s}\, a(s)\, g(kq^{n-1}(s))\, ds\right] > \frac{(n-1)!}{\gamma} \tag{17}$$

for every $k \geqslant 1$, *then the conclusion of Theorem* 1 *holds.*

In [7], Kartsatos considered Eq. (1) with $q(t)=t$, f an increasing function, and $e(t)$ satisfying condition (4), and proved that "Equation (1) is oscillatory if and only if the equation

$$x^{(n)}(t) + a(t)\, f(x(t)) = 0$$

is oscillatory." We use the above result to obtain the following corollary.

COROLLARY 3. *Let* $g(x) = 1$, $q(t) = t$ *and let condition* (5) *in Theorem* 1 *be replaced by*

$$\limsup_{t \to \infty} \left[\frac{1}{t} \int_T^t s^n a(s)\, ds + t \int_t^\infty s^{n-2} a(s)\, ds \right] > \frac{(n-1)!}{\gamma}, \tag{18}$$

then the conclusion of Theorem 1 *holds.*

Remarks. 1. From Theorem 1 and Corollary 2 we see that our results do hold for both forced equations and the associated unforced equations.

2. Our results can be applied to some cases in which results of Dahiya and Akinyele [1] and Žilina [10] are not applicable. Such cases are described in the following examples.

EXAMPLE 1. Consider the differential equations

$$x^{(3)}(t) + \frac{c}{t^\varepsilon} f(x[q(t)]) = 0, \qquad t \geqslant 1, \tag{19}$$

and

$$x^{(3)}(t) + \frac{c}{t^\varepsilon} f(x[q(t)]) = \frac{1}{t^4} [8 \cos \ln t - c \sin \ln t], \qquad t \geqslant 1, \tag{20}$$

where ε, $c > 0$. Here,

$$\eta(t) = \frac{4}{5} \frac{\sin \ln t}{t} - c \frac{\cos \ln t}{t},$$

$$e(t) = 8 \frac{\cos \ln t}{t^4} - c \frac{\sin \ln t}{t^4},$$

$$e(t) = \eta^{(3)}(t).$$

We consider the following:

(i) $f(x) = x$, $q(t) = t$, and $\varepsilon = 3$.

The conditions of Corollaries 2 and 3 are satisfied for Eqs. (19) and (20), respectively, if $c > 1$, and so every solution $x(t)$ of Eq. (19) or (20) is either oscillatory or else $x^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1, ..., n-1$.

Equation (19) has a nonoscillatory solution $x(t) = 1/\sqrt{t} \to 0$ as $t \to \infty$ and Eq. (20) has an oscillatory solution $x(t) = (\sin \ln t)/t \to 0$ as $t \to \infty$.

We observe that Eq. (20) has an oscillatory solution $x(t) = (\sin \ln t)/t$ for all $c \in (-\infty, \infty)$ and, since our results do hold for both forced equations and the associated unforced equations, then $c > 1$ is the appropriate choice to maintain the asymptotic and oscillatory properties of both Eqs. (19) and (20). Otherwise, if $c=0$, Eq. (19) has a nonoscillatory solution $x(t)=t^2$. Also for $c=-6$, the same equation has the unbounded nonoscillatory solution $x(t)=t^3$. In the last two cases, condition (17) of Corollary 2 is violated.

(ii) $$f(x)=\sinh x,\ q(t)=t+\sin t, \text{ and } \varepsilon=3.$$

Here we take $\gamma=1$ and $\sigma(t)=t-1$. Thus, all conditions of Corollary 1 are satisfied for Eqs. (19) and (20) if $c>2$ and hence every solution $x(t)$ of Eq. (19) or (20) is either oscillatory or $\lim_{t\to\infty} x^{(i)}(t)=0$, $i=0, 1, ..., n-1$.

(iii) $$f(x)=\frac{x}{1+|x|},\ q(t)=t^2, \text{ and } \varepsilon=-1.$$

Here we take $h(x)=x$, $\gamma=1$, $g(x)=1/(1+|x|)$, and $\sigma(t)=t$. The conditions of Theorem 1 are satisfied for Eqs. (19) and (20) if $c>3$ and hence every solution $x(t)$ of Eq. (19) and (20) is either oscillatory or $\lim_{t\to\infty} x^{(i)}(t)=0$, $i=0, 1, ..., n-1$.

We note that the results in [1] are not applicable to Eqs. (19) and (20) because condition (iii) of Theorem 1 and condition (iv) of Theorem 2 in [1] are violated. Also, the results in [1] hold only for linear equations with retarded or advanced arguments. Next, the results in [10] hold for unforced linear retarded equations, therefore are not applicable to forced or nonlinear equations with deviating arguments of mixed type we considered.

EXAMPLE 2. Consider the differential equation

$$x^{(3)}(t)+\frac{6}{t^3}x(t)=-\frac{2}{t^4}[3\sin\ln t+5\cos\ln t], \qquad t\geqslant 1. \tag{21}$$

Here,

$$\eta(t)=\frac{3}{5}\frac{\cos\ln t}{t}-\frac{\sin\ln t}{t}.$$

The conditions of Corollary 3 are satisfied and hence every solution $x(t)$ of Eq. (21) is either oscillatory or $\lim_{t\to\infty} x^{(i)}(t)=0$, $i=0, 1, ..., n-1$. Equation (21) has a nonoscillatory solution $x(t)=(2-\sin\ln t)/t\to 0$ as $t\to\infty$.

EXAMPLE 3. The differential equation

$$x^{(3)}(t)+\frac{c}{t^3}x(t)=\frac{c\ln t+2}{t^3}, \qquad t\geqslant 1 \tag{22}$$

has an unbounded nonoscillatory solution $x(t) = \ln t$. All conditions of Corollary 3 are satisfied for Eq. (22) if $c > 1$, except condition (4). We note that Eq. (22) has the nonoscillatory solution $x(t) = \ln t$ for all $c \in (-\infty, \infty)$, while the unforced equation

$$x^{(3)}(t) + \frac{c}{t^3} x(t) = 0, \qquad t \geqslant 1, \tag{23}$$

satisfies the conclusion of Theorem 1 if $c > 1$. Here the forcing term that appears in Eq. (22) affects the behavioral properties of Eq. (23) when $c > 1$.

EXAMPLE 4. Consider the differential equations

$$x^{(4)}(t) + \frac{c}{t^\alpha} f(x[q(t)]) = \frac{-32}{t^5} \cos \ln t - \frac{8-c}{t^5} \sin \ln t, \qquad t \geqslant 1 \tag{24}$$

and

$$x^{(4)}(t) + \frac{c}{t^\alpha} f(x[q(t)]) = 0, \qquad t \geqslant 1, \tag{25}$$

where $\alpha, c > 0$ and f and q are given below. We take

$$\eta(t) = \frac{136 - c}{170} \frac{\sin \ln t}{t} + \frac{4c}{170} \frac{\cos \ln t}{t}.$$

It is easy to check that $\eta^{(4)}(t) = e(t)$ and $\eta^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1, 2, 3$.

For Equations (24) and (25), we consider

(i) $f(x) = x$, $q(t) = t$, and $\alpha = 4$.

The conditions of Corollary 3 are satisfied for Eqs. (24) and (25) if $c > 3$ and hence all solutions of Eqs. (24) and (25) are oscillatory. One such solution of Eq. (24) is $x(t) = (\sin \ln t)/t$. Once again Eq. (24) has the oscillatory solution $x(t) = (\sin \ln t)/t$ for all $c \in (-\infty, \infty)$. Thus, we conclude that it is possible to have an even-order forced equation with oscillatory solutions and the associated unforced equation with nonoscillatory solutions. In our case, if $c = 0$ Eq. (24) has the oscillatory solution $x(t) = (\sin \ln t)/t$ and Eq. (25) has the nonoscillatory solutions t, t^2, t^3.

(ii) $f(x) = x \ln(e + x^2)$, $q(t) = t + \cos t$, and $\alpha = 4$.

We take $\gamma = 1$ and $\sigma(t) = t - 1$. All conditions of Corollary 1 are satisfied for Eqs. (24) and (25) if $c > 6$ and hence every solution of Eq. (24) or (25) is oscillatory.

(iii) $f(x) = (\sinh x)/(1 + |x|)$, $q(t) = t^{1/4}$, and $\alpha = 0$.

Here we let $h(x) = \sinh x$, $g(x) = 1/(1+|x|)$, and $\alpha(t) = t^{1/4}$. The conditions of Theorem 1 are satisfied for Eqs. (24) and (25) for all $c > 0$, and so all solutions of Eqs. (24) and (25) are oscillatory.

Because of the reasons similar to those given in Example 1, the results of [1, 10] are not applicable to Eqs. (24) and (25).

In the following theorem we require that η be bounded.

THEOREM 2. *Let condition* (4) *in Theorem* 1 *be replaced by*

$$\eta(t) \text{ is bounded and oscillatory with } \eta^{(n)}(t) = e(t) \text{ and } \eta^{(i)}(t) \to 0 \text{ as } t \to \infty,\ i = 1, 2, ..., n-1, \tag{26}$$

then for n even, Eq. (1) *is oscillatory, while for n odd, every solution* $x(t)$ *of Eq.* (1) *is either oscillatory or* $x^{(i)}(t) \to 0$ *as* $t \to \infty$, $i = 1, 2, ..., n-1$, *and* $\lim_{t\to\infty} [x(t) - \eta(t)] = 0$.

Proof. The proof is similar to that of Theorem 1 and hence is omitted.

For illustration we consider the following examples.

EXAMPLE 5. Consider the differential equation

$$\ddot{x}(t) + \frac{2}{t^2} x(t) = \frac{\sin \ln t}{t^2} - \frac{\cos \ln t}{t^2}, \qquad t \geq 1. \tag{27}$$

Now,

$$\eta(t) = \cos \ln t,$$

which is oscillatory and bounded and $\dot{\eta}(t) \to 0$ as $t \to \infty$. The hypotheses of Theorem 2 are satisfied and so every solution of Eq. (27) is oscillatory. Equation (27) has an oscillatory solution $x(t) = \sin \ln t$.

EXAMPLE 6. Consider the differential equation

$$x^{(3)}(t) + \frac{3}{t^3} x[t^\beta] = \frac{3}{t^3} \sin \ln t + \frac{1}{t^3} \cos \ln t + \frac{3}{t^3} \sin(\beta \ln t), \qquad t \geq 1, \tag{28}$$

where $\beta \geq 1$. Here we let

$$\eta(t) = \sin \ln t - \frac{3\beta^2 - 6}{\beta(\beta^2+1)(\beta^2+4)} \cos(\beta \ln t) + \frac{3}{(\beta^2+1)(\beta^2+4)} \sin(\beta \ln t),$$

and

$$\sigma(t) = t.$$

The conditions of Theorem 2 are satisfied and hence Eq. (28) is oscillatory. Equation (28) admits an oscillatory solution $x(t) = \sin \ln t$.

In the following theorem the function f is not required to be differentiable.

THEOREM 3. *Suppose that*

$$\frac{f(x)}{x} \geqslant \gamma > 0 \qquad \text{for} \quad x \neq 0, \tag{29}$$

and let the functions η *and* σ *be as in Theorem* 1. *If conditions* (3), (4), *and* (16) *are satisfied, then the conclusion of Theorem* 1 *holds.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1), say $x(t) > 0$ and $x[\sigma(t)] > 0$ for $t \geqslant t_0$. As in the proof of Theorem 1 we let $x(t) = y(t) + \eta(t)$. Then Eq. (1) becomes

$$y^{(n)}(t) + a(t)\, f(x[q(t)] = 0.$$

Also, we note that $\dot{y}(t) > 0$ for all large t and we obtain inequality (10), and hence by condition (29) we obtain

$$y^{(n)}(t) + \frac{\gamma}{2}\, a(t)\, y[\sigma(t)] \leqslant 0 \qquad \text{for all large } t.$$

The rest of the proof is similar to that of Theorem 1 and hence is omitted.

The following example is illustrative.

EXAMPLE 7. Consider the equation

$$x^{(n)}(t) + \frac{n!}{2et^n}\, x[t + \sin t]\, e^{\sin x[t + \sin t]} = \left(\frac{\sin \ln t}{t}\right)^n, \qquad t \geqslant 1. \tag{30}$$

Here $\eta(t) = (\sin \ln t)/t$ and $\sigma(t) = t - 1$. All conditions of Theorem 3 are satisfied and hence Eq. (30) is oscillatory if n is even, while every solution $x(t)$ of Eq. (30) is either oscillatory or $\lim_{t \to \infty} x^{(i)}(t) = 0$, $i = 0, 1, ..., n-1$, if n is odd.

Remark. From the proof of Theorems 1 and 3 we note that the conclusion of Theorem 1 holds even if the nondecreasing component h of the function f is replaced by any function $h\colon R \to R$ satisfying condition (2).

EXAMPLE 8. Consider the differential equation

$$x^{(n)}(t)+t^{n/2-1}\frac{x[\sqrt{t}]\,e^{\sin x[\sqrt{t}]}}{1+x^2[\sqrt{t}]}=\left(\frac{\sin \ln t}{t}\right)^n, \qquad t\geqslant 1. \tag{31}$$

Here, $\sigma(t)=\sqrt{t}$, $\eta(t)=(\sin \ln t)/t$, $g(x)=1/(1+x^2)$, $h=xe^{\sin x}$, which is not differentiable, and $\gamma=1/e$.

Now, we can apply Theorem 1 to Eq. (31) and conclude that for n even, Eq. (31) is oscillatory, while for n odd, every solution $x(t)$ of Eq. (31) is either oscillatory or $\lim_{t\to\infty} x^{(i)}(t)=0$, $i=0, 1, ..., n-1$.

The following theorem deals with the oscillatory behavior of *strongly superlinear* equations of the form of Eq. (1).

THEOREM 4. *Suppose* $f\in C(R_\alpha)$, $\alpha>0$, *and let* g *and* h *be a pair of continuous components of* f *with* h *being the nondecreasing one,*

$$h(-xy)\geqslant h(xy)\geqslant Kh(x)\,h(y), \qquad for \quad x, y>0, \tag{32}$$

where K *is a positive constant and*

$$\int^{+\infty}\frac{du}{h(u)}<\infty \qquad and \qquad \int^{-\infty}\frac{du}{h(u)}<\infty. \tag{33}$$

Suppose there exist an oscillatory function η: $[t_0, \infty)\to R$ *and a differentiable function* σ: $[t_0, \infty)\to(0, \infty)$ *such that condition* (3) *holds and*

$$\sigma(t)\leqslant\min\{t, q(t)\}, \qquad \dot\sigma(t)\geqslant 0 \qquad for \quad t\geqslant t_0 \text{ and } \lim_{t\to\infty}\sigma(t)=\infty. \tag{34}$$

If

$$\limsup_{t\to\infty}\sigma(t)\int_t^\infty s^{n-l-1}g(kq^{n-1}(s))\,h(\sigma^{l-1}(s))\,a(s)\,ds>0, \tag{35}$$

for every $k\geqslant 1$ *and* $l\in\{1, 2, ..., n-1\}$ *with* $n+l$ *odd, then for* n *even, Eq.* (1) *is oscillatory.*

Moreover, if

$$\int^\infty s^{n-1}a(s)\,ds=\infty, \tag{36}$$

then for n *odd, every solution* $x(t)$ *of Eq.* (1) *is either oscillatory or* $\lim_{t\to\infty} x^{(i)}(t)=0$, $i=0, 1, ..., n-1$.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1), say $x(t)>0$ and

$x[\sigma(t)]>0$ for $t\geqslant t_0$. As in the proof of Theorem 1 (Case 1), we consider $x(t)=y(t)+\eta(t)$ and get (11). Thus Eq. (6) becomes

$$y^{(n)}(t)\leqslant -a(t)\, g(kq^{n-1}(t))\, h(\tfrac{1}{2}y[\sigma(t)]) \qquad \text{for} \quad t\geqslant t_5. \tag{37}$$

Now, by applying Taylor's formula with remainder, we get

$$y^{(l)}(t)=\sum_{j=0}^{n-l-1}(-1)^j\frac{(s-t)^j}{j!}\, y^{(l+j)}(s)+(-1)^{n-l}\int_t^s\frac{(u-t)^{n-l-1}}{(n-l-1)!}\, y^{(n)}(u)\, du.$$

By Lemma 1 and (37) we obtain

$$y^{(l)}(t)\geqslant\int_t^\infty\frac{(u-t)^{n-l-1}}{(n-l-1)!}\, a(u)\, g(kq^{n-1}(u))\, h(\tfrac{1}{2}y[\sigma(u)])\, du. \tag{38}$$

Next, we apply Lemma 1(iii) and conclude that there exists a $t_6\geqslant t_5$ such that $\sigma(t)>t_5$ and

$$y[\sigma(t)]\geqslant\frac{[\sigma(t)-t_5]^{l-1}}{l!}\, y^{(l-1)}[\sigma(t)] \qquad \text{for all} \quad t\geqslant t_6. \tag{39}$$

Using condition (32) and inequality (39) in (38) we obtain

$$\begin{aligned} y^{(l)}(t)\geqslant M\int_t^\infty (u-t)^{n-l-1}\, a(u)\, g(kq^{n-1}(u))& \\ \times h([\sigma(u)-t_5]^{l-1})\, h(y^{(l-1)}[\sigma(u)])\, du,& \end{aligned} \tag{40}$$

where

$$M=\frac{K^2}{(n-l-1)!}\, h\left(\frac{1}{2(l!)}\right).$$

Integrating (40) from T to t, $t>T\geqslant t_6$, we get

$$\begin{aligned} y^{(l-1)}(t)\geqslant M(t-T)\int_t^\infty (u-T)^{n-l-1}\, a(u)\, g(kq^{n-1}(u))& \\ \times h([\sigma(u)-t_5]^{l-1})\, h(y^{(l-1)}[\sigma(u)])\, du.& \end{aligned}$$

Then, for all sufficiently large t,

$$\begin{aligned} \frac{1}{M}\, y^{(l-1)}[\sigma(t)]\geqslant(\sigma(t)-T)\int_t^\infty (u-T)^{n-l-1}\, a(u)\, g(kq^{n-1}(u))& \\ \times h([\sigma(u)-T]^{l-1})\, h(y^{(l-1)}[\sigma(u)])\, du.& \end{aligned}$$

Since $y^{(l-1)}(t)$ is nondecreasing for $t \geqslant T$ we have

$$\frac{1}{M}\frac{y^{(l-1)}[\sigma(t)]}{h(y^{(l-1)}[\sigma(t)])} \geqslant (\sigma(t)-T)\int_t^\infty (u-T)^{n-l-1}\, a(u) \times g(kq^{n-1}(u))\, h([\sigma(u)-T]^{l-1})\, du. \tag{41}$$

Now, as condition (33) implies that

$$\lim_{x\to\infty}\frac{x}{h(x)}=0$$

and condition (35) implies that

$$\int^\infty s^{n-l}a(u)\, g(kq^{n-1}(s))\, h(\sigma^{l-1}(s))\, ds=\infty,$$

then by Theorem 2 in [2], $y^{(l-1)}(t)$ cannot be bounded above by a constant. And hence by (41), we obtain the desired contradiction. The proof when $l=0$, i.e., n is odd, is easy and hence is omitted.

COROLLARY 4. *Let conditions* (35) *and* (36) *in Theorem* 4 *be replaced by*

$$t^m \leqslant h(\sigma^m(t)) \qquad \text{for} \quad t\geqslant T\geqslant 0 \text{ and } m=1, 2, ..., n-1, \tag{42}$$

and

$$\limsup_{t\to\infty}\sigma(t)\int_t^\infty s^{n-1}g(kq^{n-1}(s))\, a(s)\, ds>0. \tag{43}$$

Then every solution of Eq. (1) *is oscillatory if n is even and every solution $x(t)$ of Eq.* (1) *is either oscillatory or* $\lim_{t\to\infty} x^{(i)}(t)=0$, $i=0, 1, ..., n-1$, *if n is odd.*

The following example is illustrative.

EXAMPLE 9. Consider the equations

$$x^{(n)}(t)+t^{-n-1}\,|x[t+\sin t]|^\alpha \operatorname{sgn} x[t+\sin t] =\left(\frac{\sin\ln t}{t}\right)^{(n)}, \qquad t\geqslant 1,\ \alpha>1, \tag{44}$$

and

$$x^{(n)}(t) + \frac{1}{t^2} \frac{|x[t+\sin t]|^\alpha}{1 + |x[t+\sin t]|} \operatorname{sgn} x[t+\sin t] = \left(\frac{\sin \ln t}{t}\right)^{(n)}, \qquad t \geq 1, \ \alpha > 1. \tag{45}$$

Take $\sigma(t) = t - 1$ and $\eta(t) = (\sin \ln t)/t$. It is easy to check that the hypotheses of Corollary 4 are satisfied and hence, for n even, Eqs. (44) and (45) are oscillatory, while, for n odd, every solution $x(t)$ of Eq. (44) or (45) is either oscillatory or $\lim_{t \to \infty} x^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$.

We believe that none of the results in [1–10] can be applied to Eqs. (44) and (45).

References

1. R. S. Dahiya and O. Akinyele, Oscillation theorems of nth order functional differential equations with forcing terms, *J. Math. Anal. Appl.* **109** (1985), 325–332.
2. K. E. Foster and R. C. Grimmer, Nonoscillatory solutions of higher order delay equations, *J. Math. Anal. Appl.* **77** (1980), 150–164.
3. S. R. Grace and B. S. Lalli, Oscillation theorems for nth order delay differential equations, *J. Math. Anal. Appl.* **91** (1983), 352–366.
4. S. R. Grace and B. S. Lalli, On oscillation of solutions of higher order forced functional differential inequalities, *J. Math. Anal. Appl.* **91** (1983), 525–535.
5. S. R. Grace and B. S. Lalli, A comparison theorem for general nonlinear ordinary differential equations, *J. Math. Anal. Appl.*, in press.
6. A. G. Kartsatos, Recent results on oscillation of solutions of forced and perturbed nonlinear differential equations of even order, *in* "Stability of Dynamical Systems: Theory and Applications," Lecture Notes in Pure and Applied Mathematics, Vol. 28, pp. 17–72, Dekker, New York, 1977.
7. A. G. Kartsatos, The oscillation of a forced equation implies the oscillation of the unforced equation—Small forcing, *J. Math. Anal. Appl.* **76** (1980), 98–106.
8. A. G. Kartsatos, On the maintenance of oscillations under the effect of a small forcing term, *J. Differential Equations* **10** (1970), 355–363.
9. W. E. Mahfoud, Oscillation and asymptotic behavior of solutions of nth order nonlinear delay differential equations, *J. Differential Equations* **24** (1977), 75–98.
10. R. O. Žilina, Oscillation of linear retarded differential equation, *Czech. Math. J.* **34** (1984), 371–377.